

ON THE SEMINVARIANTS OF LINEAR HOMOGENEOUS
DIFFERENTIAL EQUATIONS OF THE THIRD ORDER.

by

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Let us consider the system of linear homogeneous differential equations

$$(1) \quad y_i^{(n)} + \sum_{l=0}^{n-1} \sum_{k=1}^n \binom{n}{l} p_{ikl} y_k^{(l)} = 0$$

($i = 1, 2, 3, \dots, n$)

where

$$y_k^{(l)} = \frac{d^l y_k}{dx^l}$$

and where the quantities p_{ikl} are functions of the independent variable x . Wilczynski^{*} has shown that the most general point transformation which converts this system into another of the same kind is given by the equations

$$(a) \quad y_k = \sum_{\lambda=1}^n \alpha_{k\lambda} \left(\xi \right) \eta_\lambda, \quad x = f(\xi) \quad (k = 1, 2, 3, \dots, n)$$

where $\alpha_{k\lambda}$ and f are arbitrary functions of ξ and where the determinant

$$\left| \alpha_{k\lambda} \right|, \quad (k, \lambda = 1, 2, 3, \dots, n),$$

does not vanish identically.

A function of the coefficients of (1) and of

* Wilczynski, Projective Differential Geometry of Curves and Ruled Surfaces, Leipzig, 1906, Chap. I.

their derivatives, which has the same value for (1) as for the system obtained from it by such a transformation, is called an absolute invariant. If the function contains also the dependent variables and their derivatives, it is called a covariant. Functions which remain invariant under all transformations of the above form for which the independent variable remains unaltered, so that only the dependent variables are transformed, are called seminvariants and semi-covariants.

In his Projective Differential Geometry of Curves and Ruled Surfaces, Wilczynski has calculated the seminvariants, semi-covariants, invariants, and covariants for a system of two linear homogeneous differential equations of the second order by solving systems of linear partial differential equations. In like manner, Stouffer has obtained the above invariant quantities for two linear homogeneous differential equations of the third order.* In this paper the seminvariants are found for three linear homogeneous differential equations of the third order. An effort has been made to use differential operators wherever possible. Also the

* E.B.Stouffer, Invariants of Linear Differential Equations with Applications to Ruled Surfaces in Five-dimensional Space, Proceedings of the London Mathematical Society, Ser.2, Vol.11, Part 3.

extension has been made to n linear homogeneous differential equations of the third order.

The transformations (a) give an infinite continuous group which according to Lie's terminology is defined by differential equations. Lie^{*} has shown that the invariants of such a group may always be determined as the solutions of a complete system of partial differential equations which is obtained by equating to zero the symbol of the most general infinitesimal transformation of the group.

I. Transformations of the General System.

We proceed to transform (1) by putting

$$(2) \quad y_{\kappa} = \sum_{\lambda=1}^n \alpha_{\kappa\lambda} (\xi) \eta_{\lambda}, \quad (\kappa = 1, 2, 3, \dots, n),$$

which gives a transformation of the dependent variables alone. The above considerations apply to these transformations and they form a sub-group of the infinite group defined by the transformation (a).

Let us transform (1) by means of the transformation (2) into a new system whose coefficients we

* Lie, Mathematische Annalen, 1884, Vol. XXIV, p. 537.

shall denote by $\pi_{\lambda\mu\nu}$. These new coefficients are expressed in terms of the old coefficients by the following equations *

$$(3) \quad \Delta \pi_{\lambda\mu\nu} = \sum_{i=1}^n H_{i\lambda} \left[\alpha_{i\mu}^{(m-\nu)} + \sum_{k=1}^n \sum_{l=1}^{m-i-\nu} \binom{m-\nu}{l} \beta_{i, \kappa, \nu+l} \alpha_{\kappa\mu}^{(l)} \right],$$

$$(\lambda, \mu = 1, 2, 3, \dots, n; \nu = 0, 1, 2, \dots, m-1),$$

where $H_{i\lambda}$ is the minor of $\alpha_{i\lambda}$ in the determinant

$$\Delta = \begin{vmatrix} \alpha_{i\lambda} \end{vmatrix}, \quad (i, \lambda = 1, 2, 3, \dots, n).$$

The most general infinitesimal transformation of the form (2) will be obtained by putting

$$(4) \quad \alpha_{ii} = 1 + \phi_{ii}(x) \delta t, \quad \alpha_{i\kappa} = \phi_{i\kappa}(x) \delta t,$$

$$(i \neq \kappa, i, \kappa = 1, 2, 3, \dots, n),$$

where δt is an infinitesimal and the $\phi_{i\kappa}$'s are arbitrary functions of x . We wish to find the corresponding infinitesimal transformations of the coefficients $\pi_{\lambda\mu\nu}$. They are secured by substituting these values of $\alpha_{i\kappa}$ in (3) and denoting the infinitesimal

* Wilczynski, loc.cit., p.93. In the above, the change of notation involved in the introduction of binomial coefficients in (1) accounts for the difference between (3) and the equation of Wilczynski.

difference $\pi_{\lambda\mu\nu} - \cancel{p}_{\lambda\mu\nu}$ by $\delta \cancel{p}_{\lambda\mu\nu}$. Thus

$$(5) \quad \frac{\delta \cancel{p}_{\lambda\mu\nu}}{\delta t} = \sum_{\kappa=1}^n \left(\cancel{p}_{\kappa\mu} \cancel{p}_{\lambda\kappa\nu} - \cancel{p}_{\lambda\kappa} \cancel{p}_{\kappa\mu\nu} \right) + \sum_{\kappa=1}^n \sum_{\tau=1}^{m-1-\nu} \binom{m-1-\nu}{\tau} \cancel{p}_{\kappa\mu}^{(\tau)} \cancel{p}_{\lambda, \kappa, \nu+\tau} + \cancel{p}_{\lambda\mu}^{(m-\nu)},$$

$$(\lambda, \mu = 1, 2, 3, \dots, n; \quad \nu = 0, 1, 2, \dots, m-1).$$

The infinitesimal transformations of the derivatives of $\cancel{p}_{\lambda\mu\nu}$ can be obtained from (5) directly by differentiation.

II. Calculation of the Seminvariants for $m=n=3$.

Let us confine ourselves to the special case of a system of three linear homogeneous differential equations of the third order. We may write our system of differential equations in the form

$$(6) \quad y_i''' + \sum_{\kappa=1}^3 \left[3 \cancel{p}_{i\kappa} y_{\kappa}'' + 3 q_{i\kappa} y_{\kappa}' + r_{i\kappa} y_{\kappa} \right] = 0,$$

$$(i = 1, 2, 3).$$

By comparing these equations with the general equations (1) we see that we have put

$$(7) \quad \cancel{p}_{\lambda\mu 2} = \cancel{p}_{\lambda\mu}, \quad \cancel{p}_{\lambda\mu 1} = q_{\lambda\mu}, \quad \cancel{p}_{\lambda\mu 0} = r_{\lambda\mu}.$$

The general equations (5) give for the infinitesimal transformations of the coefficients of (6) the results

$$\begin{aligned}
 (8) \quad \frac{\delta \phi_{\lambda\mu}}{\delta t} &= \sum_{k=1}^3 (\phi_{\kappa\mu} \phi_{\lambda\kappa} - \phi_{\lambda\kappa} \phi_{\kappa\mu}) + \phi'_{\lambda\mu}, \\
 \frac{\delta q_{\lambda\mu}}{\delta t} &= \sum_{k=1}^3 (\phi_{\kappa\mu} q_{\lambda\kappa} - \phi_{\lambda\kappa} q_{\kappa\mu} + 2\phi'_{\kappa\mu} \phi_{\lambda\kappa}) + \phi''_{\lambda\mu}, \\
 \frac{\delta r_{\lambda\mu}}{\delta t} &= \sum_{k=1}^3 (\phi_{\kappa\mu} r_{\lambda\kappa} - \phi_{\lambda\kappa} r_{\kappa\mu} + 3\phi'_{\kappa\mu} q_{\lambda\kappa} + 3\phi''_{\kappa\mu} \phi_{\lambda\kappa}) + \phi'''_{\lambda\mu}.
 \end{aligned}$$

Let f be any seminvariant depending only on the arguments $\phi_{\lambda\mu}, \phi'_{\lambda\mu}, q_{\lambda\mu}$. The expression

$$\delta f = \sum_{\lambda,\mu} \left(\frac{\partial f}{\partial \phi_{\lambda\mu}} \delta \phi_{\lambda\mu} + \frac{\partial f}{\partial \phi'_{\lambda\mu}} \delta \phi'_{\lambda\mu} + \frac{\partial f}{\partial q_{\lambda\mu}} \delta q_{\lambda\mu} \right),$$

which represents the increment which the infinitesimal transformation gives to f , must vanish for all values of the arbitrary functions $\phi_{rs}, \phi'_{rs}, \phi''_{rs}$. Consequently the coefficients of these 27 arbitrary functions in δf when equated to zero, gives a system of partial differential equations of which f must be a solution. Lie's theory tells us that it is a complete system, and that any solution of it is a seminvariant.

Writing out this system, we find

$$(a) \quad \frac{\partial \chi}{\partial p'_{rs}} + \frac{\partial \chi}{\partial g_{rs}} = 0,$$

$$(9) \quad (b) \quad \frac{\partial \chi}{\partial p_{rs}} = \sum_{\lambda=1}^3 \left(p_{\lambda r} \frac{\partial \chi}{\partial p_{\lambda s}} - p_{s\lambda} \frac{\partial \chi}{\partial p_{r\lambda}} + 2 p_{\lambda r} \frac{\partial \chi}{\partial g_{\lambda s}} \right) = 0,$$

$$(c) \quad \sum_{\lambda=1}^3 \left(p_{\lambda r} \frac{\partial \chi}{\partial p_{\lambda s}} - p_{s\lambda} \frac{\partial \chi}{\partial p_{r\lambda}} + p'_{\lambda r} \frac{\partial \chi}{\partial p'_{\lambda s}} - p'_{s\lambda} \frac{\partial \chi}{\partial p'_{r\lambda}} + g_{\lambda r} \frac{\partial \chi}{\partial g_{\lambda s}} - g_{s\lambda} \frac{\partial \chi}{\partial g_{r\lambda}} \right) = 0, \\ (r, s = 1, 2, 3).$$

This contains 27 independent variables and 27 equations.

But only 24 of the equations are independent, so that

there are 3 seminvariants containing only the varia-

bles $p_{\lambda\mu}, p'_{\lambda\mu}, g_{\lambda\mu}^*$.

The first nine equations of (9) show that

p'_{rs} and g_{rs} can occur only in the combinations
 $p'_{rs} - g_{rs}.$

The nine equations (b) of (9) show that $p_{rs},$

p'_{rs}, g_{rs} can occur only in the nine combinations (10)

* Horn, Einführung in die Theorie der partiellen Differentialgleichungen, Art. 5 & 6.

$$(10) \quad u_{iK} = \cancel{p'_{ik}} - q_{iK} + \sum_{j=1}^3 \cancel{p_{ij}} \cancel{p_{jk}},$$

$$(i, K = 1, 2, 3).$$

Finally, let us indicate the nine equations (c) of (9) by U_{rs} so that

$$(11) \quad U_{rs}(X) = \sum_{\lambda=1}^3 \left(\cancel{p_{\lambda r}} \frac{\cancel{\partial X}}{\cancel{\partial p_{\lambda s}}} - \cancel{p_{s\lambda}} \frac{\cancel{\partial X}}{\cancel{\partial p_{r\lambda}}} + \cancel{p'_{\lambda r}} \frac{\cancel{\partial X}}{\cancel{\partial p'_{\lambda s}}} - \cancel{p'_{s\lambda}} \frac{\cancel{\partial X}}{\cancel{\partial p'_{r\lambda}}} \right. \\ \left. + q_{\lambda r} \frac{\partial X}{\partial q_{\lambda s}} - q_{s\lambda} \frac{\partial X}{\partial q_{r\lambda}} \right) = 0.$$

$$(r, s = 1, 2, 3).$$

The one relation

$$(12) \quad U_{11} + U_{22} + U_{33} = 0.$$

is obvious.

Then finding the results $U_{ij}(U_{rs})$ we have the following table, in which the quantity in any position is obtained by operating upon the (U_{rs}) at the head of the column by the U_{ij} at the beginning of the row:

	(u_{11})	(u_{12})	(u_{13})	(u_{21})	(u_{22})	(u_{23})	(u_{31})	(u_{32})	(u_{33})
u_{11}	0	$-u_{12}$	$-u_{13}$	u_{21}	0	0	u_{31}	0	0
u_{12}	$-u_{21}$	$u_{11} - u_{22}$	$-u_{23}$	0	u_{21}	0	0	u_{31}	0
u_{13}	$-u_{31}$	$-u_{32}$	$u_{11} - u_{33}$	0	0	u_{21}	0	0	u_{31}
u_{21}	u_{12}	0	0	$u_{22} - u_{11}$	$-u_{12}$	$-u_{13}$	u_{32}	0	0
u_{22}	0	u_{12}	0	$-u_{21}$	0	$-u_{23}$	0	u_{32}	0
u_{23}	0	0	u_{12}	$-u_{31}$	$-u_{32}$	$u_{22} - u_{33}$	0	0	u_{32}
u_{31}	u_{13}	0	0	u_{23}	0	0	$u_{33} - u_{11}$	$-u_{12}$	$-u_{13}$
u_{32}	0	u_{13}	0	0	u_{23}	0	$-u_{21}$	$u_{33} - u_{22}$	$-u_{23}$
u_{33}	0	0	u_{13}	0	0	u_{23}	$-u_{31}$	$-u_{32}$	0

With the introduction of these variables (11)
 * becomes

$$u_{11}f = -u_{12} \frac{\partial f}{\partial u_{12}} - u_{13} \frac{\partial f}{\partial u_{13}} + u_{21} \frac{\partial f}{\partial u_{21}} + u_{31} \frac{\partial f}{\partial u_{31}} = 0,$$

$$(13) \quad u_{12}f = -u_{21} \frac{\partial f}{\partial u_{11}} + (u_{11} - u_{22}) \frac{\partial f}{\partial u_{12}} - u_{23} \frac{\partial f}{\partial u_{13}} + u_{21} \frac{\partial f}{\partial u_{22}} + u_{31} \frac{\partial f}{\partial u_{32}} = 0,$$

 * Horn, loc. cit., p.21-23.

$$U_{13}f = -u_{31}\frac{\partial f}{\partial u_{11}} - u_{32}\frac{\partial f}{\partial u_{12}} + (u_{11} - u_{33})\frac{\partial f}{\partial u_{13}} + u_{21}\frac{\partial f}{\partial u_{23}} + u_{31}\frac{\partial f}{\partial u_{33}} = 0,$$

$$U_{21}f = u_{12}\frac{\partial f}{\partial u_{11}} + (u_{22} - u_{11})\frac{\partial f}{\partial u_{21}} - u_{12}\frac{\partial f}{\partial u_{22}} - u_{13}\frac{\partial f}{\partial u_{23}} + u_{32}\frac{\partial f}{\partial u_{31}} = 0,$$

$$(13) \quad U_{22}f = u_{12}\frac{\partial f}{\partial u_{12}} - u_{21}\frac{\partial f}{\partial u_{21}} - u_{23}\frac{\partial f}{\partial u_{23}} + u_{32}\frac{\partial f}{\partial u_{32}} = 0,$$

$$U_{23}f = u_{12}\frac{\partial f}{\partial u_{13}} - u_{31}\frac{\partial f}{\partial u_{21}} - u_{32}\frac{\partial f}{\partial u_{22}} + (u_{22} - u_{33})\frac{\partial f}{\partial u_{23}} + u_{32}\frac{\partial f}{\partial u_{33}} = 0,$$

$$U_{31}f = u_{13}\frac{\partial f}{\partial u_{11}} + u_{23}\frac{\partial f}{\partial u_{21}} + (u_{33} - u_{11})\frac{\partial f}{\partial u_{31}} - u_{12}\frac{\partial f}{\partial u_{32}} - u_{13}\frac{\partial f}{\partial u_{33}} = 0,$$

$$U_{32}f = u_{13}\frac{\partial f}{\partial u_{12}} + u_{23}\frac{\partial f}{\partial u_{22}} - u_{21}\frac{\partial f}{\partial u_{31}} + (u_{33} - u_{22})\frac{\partial f}{\partial u_{32}} - u_{23}\frac{\partial f}{\partial u_{33}} = 0,$$

$$U_{33}f = u_{13}\frac{\partial f}{\partial u_{13}} + u_{23}\frac{\partial f}{\partial u_{23}} - u_{31}\frac{\partial f}{\partial u_{31}} - u_{32}\frac{\partial f}{\partial u_{32}} = 0.$$

The rank of the matrix formed of the above coefficients is 6, so that there are just 6 independent equations. There will, therefore, be 3 solutions for (13) and, thus, 3 seminvariants depending upon the variables $\phi_{rs}, \phi'_{rs}, q_{rs}$. They are:

$$(14) \quad I = u_{11} + u_{22} + u_{33},$$

$$J = u_{11} \cdot u_{22} \cdot u_{33} - u_{12} \cdot u_{21} \cdot u_{33} + u_{11} \cdot u_{33} \cdot u_{22} - u_{13} \cdot u_{31} \cdot u_{22} + u_{22} \cdot u_{33} \cdot u_{11} - u_{32} \cdot u_{23} \cdot u_{11} ,$$

$$K = u_{11} \cdot u_{22} \cdot u_{33} + u_{12} \cdot u_{23} \cdot u_{31} + u_{13} \cdot u_{21} \cdot u_{32} - u_{11} \cdot u_{23} \cdot u_{32} -$$

$$- u_{12} \cdot u_{21} \cdot u_{33} - u_{13} \cdot u_{31} \cdot u_{22} .$$

That these 3 seminvariants are independent is shown if we put $u_{iK} = 0$ where $i \neq K$ and notice that the functional determinant

$$\frac{\partial(I, J, K)}{\partial(u_{11}, u_{22}, u_{33})}$$

does not vanish identically.

We notice that J is obtained from K by operating on K with

$$(15) \quad \left(\frac{\partial}{\partial u_{11}} + \frac{\partial}{\partial u_{22}} + \frac{\partial}{\partial u_{33}} \right) .$$

Likewise $2I$ is obtained from J by the same operation. We shall now prove that

$$\frac{1}{r} \left(\sum_{i=1}^3 \frac{\partial}{\partial u_{ii}} \right)^r$$

operating on any seminvariant gives again a seminvariant or a constant, where r denotes the number of times

that the operator is used.

We shall need the transformations of the dependent variables. Let us, therefore, write the finite transformations of the dependent variables in the form

$$y_i = \sum_{K=1}^3 \alpha_{iK} y_K ,$$

and set

$$\left| \alpha_{iK} \right| = \Delta , \quad (i = 1, 2, 3) .$$

If we make this transformation and denote the coefficients of the new system by $\overline{p_{iK}}, \overline{g_{iK}}, \overline{r_{iK}}$, we find

$$(16) \quad \Delta \overline{p_{iK}} = \sum_{\lambda=1}^3 \overline{A_{\lambda i}} \left[\alpha'_{\lambda K} + \sum_{\mu=1}^3 \overline{p_{\lambda \mu}} \alpha_{\mu K} \right] ,$$

$$(17) \quad \Delta \overline{g_{iK}} = \sum_{\lambda=1}^3 \overline{A_{\lambda i}} \left[\alpha''_{\lambda K} + \sum_{\mu=1}^3 \overline{p_{\lambda \mu}} \alpha'_{\mu K} + \sum_{\mu=1}^3 \overline{g_{\lambda \mu}} \alpha_{\mu K} \right] ,$$

$$(18) \quad \Delta \overline{r_{iK}} = \sum_{\lambda=1}^3 \overline{A_{\lambda i}} \left[\alpha'''_{\lambda K} + \sum_{\mu=1}^3 \overline{p_{\lambda \mu}} \alpha''_{\mu K} + \sum_{\mu=1}^3 \overline{g_{\lambda \mu}} \alpha'_{\mu K} + \sum_{\mu=1}^3 \overline{r_{\lambda \mu}} \alpha_{\mu K} \right] , \quad (i, K = 1, 2, 3) ,$$

where $A_{\lambda i}$ is the minor of $\alpha_{\lambda i}$ in the determinant

$$\Delta = \begin{vmatrix} \alpha_{\lambda i} \end{vmatrix}, \quad (\lambda, i = 1, 2, 3).$$

From these equations and equations (10) we find

$$(19) \quad \Delta \overline{u_{i\kappa}} = \sum_{\lambda=1}^3 \sum_{\mu=1}^3 A_{\lambda i} \alpha_{\mu\kappa} u_{\lambda\mu}, \quad (i, \kappa = 1, 2, 3).$$

Let ϕ be a seminvariant

$$\phi(p_{i\kappa}, p'_{i\kappa}, \dots, q_{i\kappa}, q'_{i\kappa}, \dots, r_{i\kappa}, \dots),$$

which thru a transformation of the original equation becomes

$$\phi(\overline{p_{i\kappa}}, \overline{p'_{i\kappa}}, \dots, \overline{q_{i\kappa}}, \overline{q'_{i\kappa}}, \dots, \overline{r_{i\kappa}}, \dots) = \overline{\phi}.$$

Then, since ϕ is a seminvariant we must have

$$\phi(u_{i\kappa}) = \overline{\phi}(\overline{u_{i\kappa}}).$$

By the formula *

$$\frac{\partial \phi}{\partial u_{ii}} = \sum_{\lambda=1}^3 \sum_{\mu=1}^3 \frac{\partial \overline{\phi}}{\partial \overline{u_{j\kappa}}} \cdot \frac{\partial \overline{u_{j\kappa}}}{\partial u_{ii}},$$

* Goursat-Hedrick, Mathematical Analysis, Vol.1.p.58.

we obtain

$$\left(\sum_{i=1}^3 \frac{\partial}{\partial u_{ii}} \right) \phi = \left(\sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \frac{\partial \phi}{\partial u_{jk}} \cdot \frac{\partial \overline{u_{jk}}}{\partial u_{ii}} \right),$$

By the aid of (19) we have

$$\begin{aligned} \left(\sum_{i=1}^3 \frac{\partial}{\partial u_{ii}} \right) \phi &= \left(\frac{1}{\Delta} \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \frac{\partial \phi}{\partial u_{jk}} H_{ij} \alpha_{ik} \right), \\ &= \left(\frac{1}{\Delta} \sum_{j=1}^3 \sum_{k=1}^3 \frac{\partial \phi}{\partial u_{jk}} \left[\sum_{i=1}^3 H_{ij} \alpha_{ik} \right] \right). \end{aligned}$$

Since

$$\frac{1}{\Delta} H_{ij} \alpha_{ik} = 0, \quad (j \neq k),$$

and

$$\frac{1}{\Delta} H_{ij} \alpha_{ik} = 1, \quad (j = k),$$

we have

$$\left(\sum_{i=1}^3 \frac{\partial}{\partial u_{ii}} \right) \phi = \left(\sum_{i=1}^3 \frac{\partial}{\partial \overline{u_{ii}}} \right) \overline{\phi}.$$

Therefore, the operator

$$(20) \quad \left(\sum_{i=1}^3 \frac{\partial}{\partial u_{ii}} \right)$$

gives us each time a new seminvariant or a constant.

Let us find the seminvariants involving also

$$p''_{rs}, q'_{rs}, r_{rs}.$$

They must satisfy the following system of partial differential equations:

$$(a) \quad \frac{\partial X}{\partial p''_{rs}} + \frac{\partial X}{\partial q'_{rs}} + \frac{\partial X}{\partial r_{rs}} = 0,$$

$$(b) \quad \frac{\partial X}{\partial p'_{rs}} + \frac{\partial X}{\partial q_{rs}} + \sum_{\lambda=1}^3 \left(p_{\lambda r} \frac{\partial X}{\partial p''_{\lambda s}} - p_{s\lambda} \frac{\partial X}{\partial p''_{r\lambda}} + 2p_{\lambda r} \frac{\partial X}{\partial q_{\lambda s}} \right. \\ \left. + 3p_{\lambda r} \frac{\partial X}{\partial r_{\lambda s}} \right) = 0,$$

$$(21) \quad (c) \quad \frac{\partial X}{\partial p_{rs}} + \sum_{\lambda=1}^3 \left(p_{\lambda r} \frac{\partial X}{\partial p'_{\lambda s}} - p_{s\lambda} \frac{\partial X}{\partial p'_{r\lambda}} + 2p'_{\lambda r} \frac{\partial X}{\partial p''_{\lambda s}} - 2p'_{s\lambda} \frac{\partial X}{\partial p''_{r\lambda}} \right. \\ \left. + 2p_{\lambda r} \frac{\partial X}{\partial q_{\lambda s}} + 2p_{\lambda r} \frac{\partial X}{\partial q'_{\lambda s}} + q_{\lambda r} \frac{\partial X}{\partial q_{\lambda s}} - q_{s\lambda} \frac{\partial X}{\partial q_{r\lambda}} \right. \\ \left. + 3q_{\lambda r} \frac{\partial X}{\partial r_{\lambda s}} \right) = 0,$$

$$(d) \quad \sum_{\lambda=1}^3 \left(p_{\lambda r} \frac{\partial X}{\partial p_{\lambda s}} - p_{s\lambda} \frac{\partial X}{\partial p_{r\lambda}} + p'_{\lambda r} \frac{\partial X}{\partial p'_{\lambda s}} - p'_{s\lambda} \frac{\partial X}{\partial p'_{r\lambda}} + p''_{\lambda r} \frac{\partial X}{\partial p''_{\lambda s}} \right. \\ \left. - p_{s\lambda} \frac{\partial X}{\partial p''_{r\lambda}} + q_{\lambda r} \frac{\partial X}{\partial q_{\lambda s}} - q_{s\lambda} \frac{\partial X}{\partial q_{r\lambda}} + q'_{\lambda s} \frac{\partial X}{\partial q'_{\lambda s}} - q'_{s\lambda} \frac{\partial X}{\partial q'_{r\lambda}} \right. \\ \left. + r_{\lambda r} \frac{\partial X}{\partial r_{\lambda s}} - r_{s\lambda} \frac{\partial X}{\partial r_{r\lambda}} \right) = 0, \quad (r, s = 1, 2, 3).$$

There are in this system 54 variables and 36 equations. Only 35 of these equations are independent, as there is a relation here similar to the first one in the preceding case. The other relations do not persist. Therefore, there are just 19 independent seminvariants in this case.

By the method employed above we find that 27 of the equations are satisfied by the independent quantities

$$\begin{aligned}
 (22) \quad u_{iK}, \quad v_{iK} &= u'_{iK} + \sum_{j=1}^3 (\beta_{ij} u_{jK} - \beta_{jK} u_{ij}), \\
 t_{iK} &= \beta''_{iK} - 3q'_{iK} + 2r_{iK} - \sum_{j=1}^3 (\beta_{ij} \beta'_{jK} - 3\beta_{jK} \beta_{ij} \\
 &\quad + 2\beta_{jK} q_{ij} - 3\beta_{ij} u_{jK} - \beta_{jK} u_{ij}), \\
 &\quad (i, K = 1, 2, 3).
 \end{aligned}$$

The quantities u_{iK}, v_{iK}, t_{iK} are cogredient under the transformation of the dependent variables. This fact is made evident by the infinitesimal transformation of these quantities

$$(23) \quad \frac{\delta u_{iK}}{\delta t} = \sum_{j=1}^3 (u_{ij} \phi_{jK} - u_{jK} \phi_{ij}),$$

$$(23) \quad \frac{\delta v_{ik}}{\delta t} = \sum_{j=1}^3 (v_{ij} \phi_{jk} - v_{jk} \phi_{ij}) ,$$

$$\frac{\delta t_{ik}}{\delta t} = \sum_{j=1}^3 (t_{ij} \phi_{jk} - t_{jk} \phi_{ij}) .$$

The remaining 8 equations of (21) are satisfied by 19 combinations of the quantities of (22). Of these we know already I, I', J, J', K, K' . In order to obtain the remaining seminvariants let us make use of an operator of the form

$$(24) \quad \frac{1}{r} \left(v_{ik} \frac{\partial}{\partial u_{ik}} \right)^r .$$

That this operator each time gives us another seminvariant will now be shown. As above, let ϕ be a seminvariant so that

$$\overline{\phi(u_{ik})} = \phi(u_{ik}) .$$

Similarly if we have a second set of expressions v_{ik} where v_{ik} is transformed cogrediently with u_{ik} , the same function of v_{ik} will be a seminvariant. Thus, we have

$$\overline{\phi(v_{ik})} = \phi(v_{ik}) .$$

Then since the transformations of the $u_{iK}'s$ and $v_{iK}'s$ are a linear combination of the $u_{iK}'s$ and $v_{iK}'s$ a linear combination of $u_{iK}'s$ and $v_{iK}'s$ is transformed cogrediently with u_{iK} and v_{iK} . Thus we have

$$\bar{\phi}(\bar{u}_{iK} + \lambda v_{iK}) = \phi(u_{iK} + \lambda v_{iK})$$

for all values of λ .

Therefore, expanding by Taylor's Theorem and equating coefficients of λ we have

$$\left(\bar{v}_{iK} \frac{\partial}{\partial \bar{u}_{iK}} \right) \bar{\phi} = \left(v_{iK} \frac{\partial}{\partial u_{iK}} \right) \phi.$$

Hence, the operator (24) gives again a seminvariant.

This operator is analogous to the Aronhold operator. * The importance of the Aronhold operator lies in the fact that it enables us to construct simultaneous invariants or covariants of several algebraic forms of the same order when any invariants or covariants are known for a form of that order.

Likewise, since \bar{u}_{iK} is transformed cogrediently with u_{iK} and v_{iK} we can use the op-

* Grace and Young, Algebra of Invariants, § 35.

erator

$$(25) \quad \frac{1}{r} \left(t_{ik} \frac{\partial}{\partial u_{ik}} \right)^r$$

to obtain seminvariants from a given seminvariant.

The operator (24) operating upon I, J, K , gives us 6 more seminvariants. Nine more may be obtained by the use of the operator (25) upon the nine seminvariants found above.

Let us now explain our mode of representation for the 18 seminvariants already found. Since all the seminvariants may be obtained from K by means of the operators, we shall denote it by $\Delta_{(000)}$. Then let us represent any seminvariant which has been obtained from $\Delta_{(000)}$ by operating r time with the operator (20) by $\Delta_{(r00)}$, ($r = 1, 2, 3$). Thus

$\Delta_{(100)}$ represents the seminvariant obtained by the use of the operator once upon $\Delta_{(000)}$, $[\Delta_{(100)} = J]$ and

$\Delta_{(200)}$ denotes the seminvariant found by its use a second time upon $\Delta_{(000)}$, $[\Delta_{(200)} = I]$.

Similarly, let us denote any seminvariant which has been obtained from $\Delta_{(000)}$ by means of the operator (24) by $\Delta_{(0r0)}$, ($r = 1, 2, 3$), so that $\Delta_{(010)}$

denotes the seminvariant obtained by the use of the operator once upon $\Delta_{(ooo)}$ and $\Delta_{(o2o)}$ the seminvariant found by its use a second time. Likewise, we can represent any seminvariant obtained by the use of the operator (25) by $\Delta_{(oor)}$, ($r = 1, 2, 3$). Where any seminvariant is obtained by the use of two or more of the above operators operating on $\Delta_{(ooo)}$ it is denoted by the use of subscripts in the proper position in the parenthesis to indicate the number of times each operator has been used. Thus, $\Delta_{(11o)}$ represents the use of the operators (24) and (20) on $\Delta_{(ooo)}$, each one operating once, etc.

Using this mode of representation the seminvariants so far found are:

$$\begin{aligned}
 &\Delta_{(ooo)}, \Delta_{(1oo)}, \Delta_{(2oo)}, \\
 &\Delta_{(o1o)}, \Delta_{(o2o)}, \Delta_{(o3o)}, \Delta_{(o3o)}, \Delta_{(23o)}, \\
 &\Delta_{(oo1)}, \Delta_{(oo2)}, \Delta_{(oo3)}, \Delta_{(1o3)}, \Delta_{(2o3)}, \\
 &\Delta_{(o31)}, \Delta_{(o32)}, \\
 &\Delta_{(11o)}, \Delta_{(1o1)}, \Delta_{(131)}.
 \end{aligned}$$

It is left to find the one remaining seminvar-

inant. Let us first prove that V_{ij} is transformed cogrediently with V_{ji} where V_{ij} is the algebraic minor of V_{ij} in the determinant

$$\begin{vmatrix} V_{ii} & V_{ij} & V_{iK} \\ V_{ji} & V_{jj} & V_{jK} \\ V_{Ki} & V_{Kj} & V_{KK} \end{vmatrix}$$

Substituting the infinitesimal transformations of the

$$V'S, \quad \frac{\delta V_{ij}}{\delta t} \quad \text{becomes}$$

$$\sum_{\mu=1}^3 \left[V_{j\mu} \begin{vmatrix} V_{KK} & \phi_{\mu K} \\ V_{Ki} & \phi_{\mu i} \end{vmatrix} - V_{\mu i} \begin{vmatrix} V_{KK} & \phi_{K\mu} \\ V_{jK} & \phi_{j\mu} \end{vmatrix} + V_{K\mu} \begin{vmatrix} V_{ji} & \phi_{\mu i} \\ V_{jK} & \phi_{\mu K} \end{vmatrix} - V_{\mu K} \begin{vmatrix} V_{ji} & \phi_{j\mu} \\ V_{Ki} & \phi_{K\mu} \end{vmatrix} \right],$$

which gives us

$$\frac{\delta V_{ij}}{\delta t} = \sum_{\mu=1}^3 (V_{j\mu} \phi_{\mu i} - V_{\mu i} \phi_{j\mu}).$$

Therefore, V_{ij} is transformed cogrediently with V_{ji} .

Let us now make use of a new operator

$$(26) \quad \frac{1}{r} \left(V_{ij} \frac{\partial}{\partial u_{ji}} \right)^r.$$

That this operator each time gives us another seminvar-

inant is shown by a proof similar to that used for the operator (24). Let us denote the seminvariant obtained by operating upon $\Delta_{(000)}$ with this operator by $\Delta_{(0001)}$.

We have now obtained 19 seminvariants. It should be noted that in this list

$$\begin{aligned}\Delta_{(230)} &= \Delta'_{(200)}, \\ \Delta_{(010)} &= \Delta'_{(000)}, \\ \Delta_{(110)} &= \Delta'_{(100)}.\end{aligned}$$

That these 19 seminvariants are independent is seen if we put $u_{iK} = 0$, ($i \neq K$), and notice that the functional determinant

$$\partial \left(\begin{array}{cccccccccccccccc} \Delta_{(000)}, \Delta_{(100)}, \Delta_{(200)}, \Delta_{(230)}, \Delta_{(030)}, \Delta_{(203)}, \Delta_{(130)}, \Delta_{(103)}, \Delta_{(003)}, \\ \Delta_{(010)}, \Delta_{(020)}, \Delta_{(001)}, \Delta_{(002)}, \Delta_{(003)}, \Delta_{(032)}, \Delta_{(110)}, \Delta_{(101)}, \Delta_{(131)}, \Delta_{(0001)} \end{array} \right)$$

$$\partial (u_{11}, u_{22}, u_{33}, v_{11}, v_{22}, v_{33}, v_{12}, v_{21}, v_{23}, v_{32}, v_{31}, v_{13}, t_{11}, t_{12}, t_{13}, t_{21}, t_{22}, t_{23}, t_{31}, t_{32}, t_{33})$$

does not vanish identically. The above determinant is the product of a sixteenth order determinant and a third order determinant

$$\begin{vmatrix} 1 & 1 & 1 \\ u_{22} \cdot u_{33} & u_{11} \cdot u_{33} & u_{11} \cdot u_{22} \\ u_{22} + u_{33} & u_{11} + u_{33} & u_{11} + u_{22} \end{vmatrix}$$

which is different from zero. Again, the sixteenth order determinant is the product of a thirteenth order determinant and the same third order determinant. Likewise, the thirteenth order determinant is the product of a tenth order determinant and the same third order determinant. Finally, the tenth order determinant is the product of two fifth order determinants which do not vanish identically. Thus, the 19 seminvariants are independent.

We might now write down the differential equations satisfied by the seminvariants involving, besides the quantities already considered, also $\phi''_{rs}, q''_{rs}, r'_{rs}$. We should find a system of 45 equations with the one relation maintaining itself, and 81 independent variables. Thus, there are 37 seminvariants or 18 new seminvariants. Now 16 of these are

$$\begin{aligned} &\Delta''_{(200)}, \Delta'_{(203)}, \Delta'_{(100)}, \Delta'_{(130)}, \Delta'_{(103)}, \Delta''_{(000)}, \\ &\Delta'_{(030)}, \Delta'_{(003)}, \Delta'_{(020)}, \Delta'_{(001)}, \Delta'_{(002)}, \Delta'_{(031)}, \\ &\Delta'_{(032)}, \Delta'_{(101)}, \Delta'_{(131)}, \Delta'_{(0001)}, \end{aligned}$$

which are obviously independent, since they are the de-

rivatives of independent seminvariants.

We may obtain the remaining two seminvariants without writing down and integrating the last mentioned 45 equations. The quantities u_{iK} , v_{iK} , t_{iK} are cogredient under a transformation of the dependent variables. This fact has been made evident in (23).

Now certain combinations of the $u_{iK}'s$ and the $\phi_{iK}'s$ we found to be seminvariants. If we replace u_{iK} by v_{iK} , or by t_{iK} in these combinations, we must still have seminvariants. Let us form the quantities w_{iK} from the $v_{iK}'s$ in the same way that the latter are formed from the $u_{iK}'s$, so that we have

$$(27) \quad w_{iK} = v_{iK}' + \sum_{j=1}^3 (\phi_{ij}' v_{jK} - \phi_{jK}' v_{ij}).$$

Of course, the $w_{iK}'s$ are cogredient with the $u_{iK}'s$, $v_{iK}'s$ and $t_{iK}'s$ under a transformation of the dependent variables. Since w_{iK} is cogredient with u_{iK} let us form a new operator

$$(29) \quad \frac{1}{r} \left(w_{iK} \frac{\partial}{\partial u_{iK}} \right)^r,$$

which will give each time a new seminvariant, as the proof which has been used for the operators (24) and (25) holds for this. Denoting any seminvariant which can be obtained from $\Delta_{(000)}$ by means of the operator (29) by $\Delta_{(0000r)}$, ($r=1,2,3$), where r indicates the number of times the operator has been used, we find two seminvariants which are independent to be

$\Delta_{(00003)}$, $\Delta_{(10003)}$. That they are independent of the other seminvariants and of their first derivatives can easily be seen if we put $u_{ik}=0$, $v_{ik}=0$, $t_{ik}=0$, and notice that all the seminvariants vanish but

$$\Delta_{(00003)}, \Delta_{(10003)} \quad \text{in the } \omega's.$$

Thus we have found 37 independent seminvariants involving only

$$p_{ik}, p'_{ik}, p''_{ik}, p'''_{ik}, q_{ik}, q'_{ik}, q''_{ik}, r_{ik}, r'_{ik}.$$

Any other seminvariant depending on these quantities alone must be expressed in terms of the seminvariants which have been already found.

The system of equations for the seminvariants involving also the next higher derivatives of p_{ik}, q_{ik}, r_{ik} contains more independent equations and 27 more variables than in the last case. Therefore,

there exist 18 more seminvariants.

These are

evidently the next higher derivatives of

$$\begin{aligned} &\Delta''_{(200)}, \Delta'_{(203)}, \Delta'_{(100)}, \Delta'_{(130)}, \Delta'_{(103)}, \Delta''_{(000)}, \Delta'_{(003)}, \\ &\Delta'_{(020)}, \Delta'_{(001)}, \Delta'_{(002)}, \Delta'_{(031)}, \Delta'_{(032)}, \Delta'_{(101)}, \Delta'_{(131)} \\ &\Delta_{(0001)}, \Delta_{(10003)}, \Delta_{(00003)}, \Delta'_{(030)}, \end{aligned}$$

and are certainly independent.

proceeding in this

way, each step introduces 18 new seminvariants but

these are clearly obtained by performing an additional

differentiation upon those found above.

Thus, all

seminvariants, of the system (14), are functions of

$$\begin{aligned} &\Delta_{(200)}, \Delta_{(100)}, \Delta_{(203)}, \Delta_{(103)}, \Delta_{(130)}, \Delta_{(000)}, \Delta_{(030)}, \\ &\Delta_{(003)}, \Delta_{(020)}, \Delta_{(001)}, \Delta_{(002)}, \Delta_{(031)}, \Delta_{(032)}, \Delta_{(101)}, \\ &\Delta_{(131)}, \Delta_{(0001)}, \Delta_{(10003)}, \Delta_{(00003)}, \end{aligned}$$

and their derivatives.

III. Calculation of the Seminvariants for

$$m = 3, \text{ and } n = n.$$

Let us now consider the case of a system of n linear homogeneous differential equations of the third order. We may write our system of equations in the form

$$(30) \quad y_i''' + \sum_{k=1}^n \left[3p_{ik} y_k'' + 3q_{ik} y_k' + r_{ik} y_k \right] = 0, \\ (i = 1, 2, 3, \dots, n).$$

By comparing these equations with the general equations (1) we see that we have put

$$p_{\lambda\mu 2} = p_{\lambda\mu}, \quad p_{\lambda\mu 1} = q_{\lambda\mu}, \quad p_{\lambda\mu 0} = r_{\lambda\mu}.$$

The general equations (5) give for the infinitesimal transformations of the coefficients of (30) the results

$$(31) \quad \frac{\delta p_{\lambda\mu}}{\delta t} = \sum_{k=1}^n (\phi_{k\mu} p_{\lambda k} - \phi_{\lambda k} p_{k\mu}) + \phi'_{\lambda\mu}, \\ \frac{\delta q_{\lambda\mu}}{\delta t} = \sum_{k=1}^n (\phi_{k\mu} q_{\lambda k} - \phi_{\lambda k} q_{k\mu} + 2\phi'_{k\mu} p_{\lambda k}) + \phi''_{\lambda\mu}, \\ \frac{\delta r_{\lambda\mu}}{\delta t} = \sum_{k=1}^n (\phi_{\lambda\mu} r_{\lambda k} - \phi_{\lambda k} r_{k\mu} + 3\phi'_{k\mu} q_{\lambda k} + 3\phi''_{k\mu} p_{\lambda k}) + \phi'''_{\lambda\mu}.$$

Let χ be any seminvariant depending only on the arguments $p_{\lambda\mu}, p'_{\lambda\mu}, q_{\lambda\mu}$. The expression

$$\delta\chi = \sum_{\lambda,\mu=1}^n \left(\frac{\partial\chi}{\partial p_{\lambda\mu}} \delta p_{\lambda\mu} + \frac{\partial\chi}{\partial p'_{\lambda\mu}} \delta p'_{\lambda\mu} + \frac{\partial\chi}{\partial q_{\lambda\mu}} \delta q_{\lambda\mu} \right),$$

which represents the increment which the infinitesimal transformations give to χ , must vanish for all values of the arbitrary functions $\phi_{rs}, \phi'_{rs}, \phi''_{rs}$. Consequently, the coefficients of these $3n^2$ arbitrary functions in $\delta\chi$ when equated to zero, gives a system of partial differential equations of which χ must be a solution. Lie's theory tells us that it is a complete system, and that any solution of it is a seminvariant.

Writing out this system, we find

$$(a) \quad \frac{\partial\chi}{\partial p'_{rs}} + \frac{\partial\chi}{\partial q_{rs}} = 0,$$

$$(32) \quad (b) \quad \frac{\partial\chi}{\partial p_{rs}} = \sum_{\lambda=1}^n \left(p_{\lambda r} \frac{\partial\chi}{\partial p_{\lambda s}} - p_{s\lambda} \frac{\partial\chi}{\partial p_{r\lambda}} + 2p_{\lambda r} \frac{\partial\chi}{\partial q_{\lambda s}} \right) = 0,$$

$$(c) \quad \sum_{\lambda=1}^n \left(p_{\lambda r} \frac{\partial\chi}{\partial p_{\lambda s}} - p_{s\lambda} \frac{\partial\chi}{\partial p_{r\lambda}} + p'_{\lambda r} \frac{\partial\chi}{\partial p'_{\lambda s}} - p'_{s\lambda} \frac{\partial\chi}{\partial p'_{r\lambda}} + \right.$$

$$\left(g_{\lambda n} \frac{\partial f}{\partial g_{\lambda s}} - g_{s\lambda} \frac{\partial f}{\partial g_{n\lambda}} \right) = 0, \quad (n, s = 1, 2, 3, \dots, n).$$

This contains $3n^2$ independent variables and $3n^2$ equations. But only $n(3n-1)$ of the equations are independent. One relation between them is evident.

For, the sum of the left members of the equation of (32) (c), obtained by putting $r=s=i$, ($i=1, 2, 3, \dots, n$), is zero.

We shall see that there are $n-1$ other relations between the equations of (32) (c). There are, therefore, n seminvariants containing only the variables $\phi_{\lambda\mu}, \phi'_{\lambda\mu}, g_{\lambda\mu}$.

The first n^2 equations (a) of (32) show that ϕ'_{ns} and g_{ns} can occur only in the combinations $\phi'_{ns} - g_{ns}$.

The next n^2 equations (b) show that $\phi_{ns}, \phi'_{ns}, g_{ns}$ can occur only in the n^2 combinations

$$(33) \quad u_{ik} = \phi'_{ik} - g_{ik} + \sum_{j=1}^n \phi_{ij} \phi_{jk}, \quad (i, k = 1, 2, 3, \dots, n).$$

Finally, let us indicate the n^2 equations of (c) by U_{rs} so that

$$(34) \quad \mathcal{U}_{rs}(X) = \sum_{\lambda=1}^n \left(p_{\lambda n} \frac{\partial X}{\partial p_{\lambda s}} - p_{s\lambda} \frac{\partial X}{\partial p_{n\lambda}} + p'_{\lambda n} \frac{\partial X}{\partial p'_{\lambda s}} - p'_{s\lambda} \frac{\partial X}{\partial p'_{n\lambda}} \right. \\ \left. + q_{\lambda n} \frac{\partial X}{\partial q_{\lambda s}} - q_{s\lambda} \frac{\partial X}{\partial q_{n\lambda}} \right) = 0, \\ (n, s = 1, 2, 3, \dots, n).$$

The one relation

$$(35) \quad \sum_{i=1}^n \mathcal{U}_{ii} = 0$$

is obvious.

Then finding the results $\mathcal{U}_{rs}(u_{rs})$ we have by the introduction of the variables just found that (34) becomes a new system of n^2 partial differential equations in which the matrix of the coefficients is of rank $n(3n-1)$. There are, therefore, $n(3n-1)$ independent equations, so that there are n solutions and thus n seminvariants depending upon the variables p_{ns}, p'_{ns}, q_{rs} . They might be obtained by solving the system of partial differential equations but since in the case just considered we were able to find all the seminvariants from one by means of the operators given, let us prove that

$$(36) \quad \begin{vmatrix} u_{11} & u_{12} & \cdots & u_{1k} & \cdots & u_{1n} \\ u_{21} & u_{22} & \cdots & u_{2k} & \cdots & u_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ u_{k1} & u_{k2} & \cdots & u_{kk} & \cdots & u_{kn} \\ \vdots & \vdots & & \vdots & & \vdots \\ u_{n1} & u_{n2} & \cdots & u_{nk} & \cdots & u_{nn} \end{vmatrix} = \Delta_{(000)}$$

is a seminvariant. We have from (23) the infinitesimal transformations of the dependent variables which if substituted in the above determinant gives

$$(37) \quad \sum_{K=1}^n \begin{vmatrix} u_{11} & u_{12} & \cdots & \sum_{j=1}^n (u_{1j} \phi_{jk} - u_{jk} \phi_{1j}) & \cdots \\ u_{21} & \cdots & \cdots & \sum_{j=1}^n (u_{2j} \phi_{jk} - u_{jk} \phi_{2j}) & \cdots \\ \vdots & & & \vdots & \\ u_{k1} & \cdots & \cdots & \sum_{j=1}^n (u_{kj} \phi_{jk} - u_{jk} \phi_{kj}) & \cdots \\ u_{n1} & \cdots & \cdots & \sum_{j=1}^n (u_{nj} \phi_{jk} - u_{jk} \phi_{nj}) & \cdots \end{vmatrix}.$$

This sum may be written in the form

$$(38) \quad \sum_{K=1}^n \left[\Delta_{(000)} \phi_{KK} - \sum_{i=1}^n \sum_{j=1}^n u_{jk} \phi_{ij} u_{ik} \right],$$

where u_{ik} is the minor of u_{ik} in $\Delta_{(000)}$.

Then since

$$\sum_{k=1}^n \sum_{i=1}^n \sum_{j=1}^n u_{jk} u_{ik} = 0, \quad (i \neq j),$$

(38) becomes

$$(39) \quad \sum_{k=1}^n \left[\Delta_{(000)} \phi_{kk} - \sum_{i=1}^n u_{ik} \phi_{ii} u_{ik} \right].$$

Also

$$\sum_{k=1}^n \sum_{i=1}^n u_{ik} u_{ik} = \Delta_{(000)}.$$

Then (39) becomes

$$\sum_{k=1}^n \Delta_{(000)} \phi_{kk} - \sum_{i=1}^n \Delta_{(000)} \phi_{ii} = 0.$$

Therefore, the determinant $\Delta_{(000)}$ is a seminvariant.

By a proof similar to that used in showing that the operator (20) each time gave a seminvariant or a constant we can show that the operator,

$$(40) \quad \frac{1}{n} \left(\sum_{i=1}^n \frac{\partial}{\partial u_{ii}} \right)^n,$$

gives a seminvariant or a constant. Thus by operating upon $\Delta_{(000)}$ (n-1) times we obtain (n-1) seminvariants which are independent.

Let us next find the seminvariants involving also $\phi''_{rs}, g'_{rs}, r_{rs}$. They must satisfy a system of $4n^2$ partial differential equations similar to those of (21) except that the summations are from 1 to n instead of from 1 to 3 as in (21). There are in this system $6n^2$ variables and $4n^2$ equations. Since a relation similar to the first in the preceding case persists while the others do not, we have $4n^2 - 1$ equations which are independent. There are, therefore, $2n^2 + 1$ independent seminvariants in this case.

By the method employed above we ~~should~~ find that $3n^2$ of the equations ~~were~~ ^{are} satisfied by the independent quantities

$$\begin{aligned}
 u_{ik}, \quad v_{ik} &= u'_{ik} + \sum_{j=1}^n (\phi_{ij} u_{jk} - \phi_{jk} u_{ij}), \\
 (42) \quad t_{ik} &= \phi''_{ik} - 3g'_{ik} + 2r_{ik} - \sum_{j=1}^n (\phi_{ij}\phi'_{jk} - 3\phi_{jk}\phi'_{ij} \\
 &\quad + 2\phi_{jk}g'_{ij} - 3\phi_{ij}u_{jk} - \phi_{jk}u_{ij}), \\
 &\quad (i, k = 1, 2, 3, \dots, n).
 \end{aligned}$$

The quantities u_{ik}, v_{ik}, t_{ik} are cogredient under the transformations of the dependent variables as was shown in (23).

The remaining n^2-1 equations are satisfied by $2n^2+1$ combinations of the quantities of (42). Of these we know already $\Delta_{(000)}$, the $n-1$ seminvariants found by means of the operator (40), and the first derivatives of each, making in all $2n$. In order to obtain the remaining $(2n^2-n+1)$ seminvariants let us make use of the operator (24) which we have shown gives us each time a seminvariant. Operating on the n seminvariants already found by this operator we obtain $\frac{n}{2}(n+1)$ seminvariants.

Next by means of the operator (25) which we have proved gives a seminvariant each time, operating upon the seminvariants already found we obtain n^2 other seminvariants. In all, by the operators so far used we have found $\frac{3n^2}{2} + \frac{3n}{2}$ seminvariants.

It remains to find $\frac{n^2}{2} - \frac{3n}{2} + 1$ other seminvariants

by means of operators (26) which we have previously proved gives a seminvariant and others which can be shown to give each time a seminvariant by a proof similar to that used for (26). This will give us the

seminvariants for $\phi_{rs}, \phi'_{rs}, \phi''_{rs}, q_{rs}, q'_{rs}, r_{rs}$. That these $2n^2+1$ seminvariants are independent

will be left an open question.

We might now write the differential equations satisfied by the seminvariants involving, besides the quantities already considered, also $\beta_{rs}''', \beta_{rs}'', r_{rs}'$. We should find a system of $5n^2$ equations with the one relation maintaining itself, and $9n^2$ independent variables. Thus there are $4n^2+1$ seminvariants or $2n^2$ new seminvariants. Now $2n^2-n+1$ of these are the derivatives of previously found seminvariants which are obviously independent. Therefore, the number of the new seminvariants is $n-1$.

We may obtain the $n-1$ seminvariants without writing down and integrating the last mentioned $5n^2$ equations. The quantities u_{iK}, v_{iK}, t_{iK} are cogredient under the transformation of the dependent variables as was made evident in (23). Now certain combinations of the u_{iK} 's and the β_{iK} 's we found to be seminvariants. If we replace u_{iK} by v_{iK} or by t_{iK} in these combinations, we must still have seminvariants. Let us form the quantities w_{iK} from the v_{iK} 's in the same way that the latter are formed from the u_{iK} 's, so that we have

$$(42) \quad \omega_{iK} = v_{iK}' + \sum_{j=1}^n (\phi_{ij} v_{jK} - \phi_{jK} v_{ij}).$$

Of course, the ω_{iK}' s are cogredient with the u_{iK}' s, v_{iK}' s and t_{iK}' s under a transformation of the dependent variables. Since ω_{iK} is cogredient with u_{iK} let us form the operator (29) which we have previously proved will each time give a new seminvariant. This operator will give us the required new seminvariants. Thus, we have now found $4n^2+1$ seminvariants involving only

$$\phi_{rs}, \phi_{rs}', \phi_{rs}'', \phi_{rs}''', g_{rs}, g_{rs}', g_{rs}'', r_{rs}, r_{rs}'.$$

Any other seminvariant depending on these quantities above must be expressed in terms of the seminvariants which have already been found.

The system of equations for the seminvariants involving also the next higher derivatives of $\phi_{rs}, g_{rs}, r_{rs}$, contains n^2 more independent equations and $3n^2$ more variables than in the last case. Therefore, there exist $2n^2$ more seminvariants. These are evidently the next higher derivatives of those above and are certainly independent if those above are independent. Proceeding in this way, each step

introduces $2n^2$ new seminvariants but these are clearly obtained by performing an additional differentiation upon those found above.